Written Exam for the M.Sc. in Economics, Winter 2010/2011

ADVANCED MACROECONOMETRICS

Final Exam
January 26, 10:00 – January 28, 10:00

PLEASE NOTE that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish. If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

FOCUS ON EXAM CHEATING: In case of presumed exam cheating, which is observed by either the examination registration of the respective study programmes, the invigilation or the course lecturer, the Head of Studies will make a preliminary inquiry into the matter, requesting a statement from the course lecturer and possibly the invigilation, too. Furthermore, the Head of Studies will interview the student. If the Head of Studies finds that there are reasonable grounds to suspect exam cheating, the issue will be reported to the Rector. In the course of the study and during examinations, the student is expected to conform to the rules and regulations governing academic integrity. Academic dishonesty includes falsification, plagiarism, failure to disclose information, and any other kind of misrepresentation of the student's own performance and results or assisting another student herewith. For example failure to indicate sources in written assignments is regarded as failure to disclose information. Attempts to cheat at examinations are dealt with in the same manner as exam cheating which has been carried through. In case of exam cheating, the following sanctions may be imposed by the Rector:

1. A warning
2. Expulsion from the examination
3. Suspension from the University for at limited period or permanent expulsion.

The Faculty of Social Sciences
The Study and Examination Office
October 2006

Practical Information

Note the following formal requirements:

- This is an *individual* examination. You are not allowed to cooperate with other students or other people, see the *focus on exam cheating* above.
- The assignment consists of Sections 1-6 with 22 questions to be answered. *Please answer all questions.*
- The exam paper should not exceed 20 pages. A maximum of 20 pages of supporting material (graphs, estimation output, etc.) can accompany the paper as appendices. You may refer to the computer output in the appendices when answering the questions. Also, you may add clarifying comments in the output as part of your answer.
- All pages must be numbered consecutively and marked with your *exam number*. You should *not* write your name on the exam paper.
- *Your paper must be uploaded on the course page in Absalon at the given time.* The exam paper (including supporting material) must be in *PDF-format* and collected in *one file only*; the uploaded file must be named `1234.pdf`, where `1234` is your exam number.

Regarding the data for the exam paper, please note the following:

- All assignments are based on *different* data sets. You should use the data set located in the Excel file `Data1234.xls`, where `1234` is your exam number.
- To avoid that some data sets are more difficult to handle than others, the data sets are artificial (simulated from a known data generating process), and they behave, as close as possible, like actual data.

1 Background

The topic for this project examination is *term structure modelling*, i.e. the analysis of the relationship between interest rates with different maturities. The purpose of the examination is to assess your understanding of the cointegrated VAR (CVAR) model, your ability to use statistical procedures to make inference on the equilibrium structures and the dynamic adjustment properties, as well as your ability to interpret the results. Most questions in the examination are applied, concerning the empirical example outlined below. When you answer these empirical questions, please explain and motivate your answer as detailed as possible, preferably with reference to the underlying theory.

The data set you are given consists of five interest rates: $R0_t$ is a very short maturity (two-weeks) interest rates, while $R1_t$, $R2_t$, $R5_t$, and $R10_t$ measure the yield on bonds with maturities of 1, 2, 5, and 10 years, respectively. All time series are returns from year to year in percentages and they are recorded monthly for the period January 1985 to September 2010. The empirical analysis is based on the $p = 5$ dimensional data vector,

$$x_t = (R0_t : R1_t : R2_t : R5_t : R10_t)', \tag{1.1}$$

and we want to use the CVAR model as the statistical framework.

[1] Consider a $p-$dimensional data vector, x_t, and a CVAR with $k = 1$ lags:

$$\Delta x_t = \alpha\beta' x_{t-1} + \epsilon_t, \quad t = 1, 2, ..., T, \tag{1.2}$$

with $x_0 = 0$ given. Assume that the characteristic polynomial of the model has exactly $p - r$ unit roots and that the remaining roots are in the stationary region. Derive the Granger representation of the model in (1.2), i.e. the solution of x_t in terms of the sequence of current and past innovations, $\epsilon_t, \epsilon_{t-1}, ..., \epsilon_1$. Explain the concept of pulling and pushing forces.

One simple theoretical approach to modelling the different interest rates suggests that all interest rates are driven by a single underlying *factor*. Presuming the presence of unit roots in the interest rate time series, we will interpret the factor as a common stochastic trend. Using the Granger representation of the CVAR, the *one-factor model* predicts the following scenario for interest rates

$$\begin{pmatrix} R0_t \\ R1_t \\ R2_t \\ R5_t \\ R10_t \end{pmatrix} = \begin{pmatrix} 1 \\ \tau_1 \\ \tau_2 \\ \tau_5 \\ \tau_{10} \end{pmatrix} \left(\textstyle\sum_{i=1}^{t} u_{1i}\right) + \text{stationary process} + \text{initial value}, \tag{1.3}$$

where the factor, $f_t^1 = \sum_{i=1}^{t} u_{1i}$, is an I(1) stochastic trend.

[2] Explain what the scenario in (1.3) implies in terms of possible cointegrating relations between the interest rates. More specifically, you should specify the cointegration rank of the corresponding CVAR and the implied structure for the cointegration space, i.e. a matrix β so that $\beta' x_t$ is a stationary process.
Explain why the choice of cointegration matrix, β, is not unique in the CVAR model. Finally, explain how the results are modified if $\tau_1 = \tau_2 = \tau_5 = \tau_{10} = 1$.

A more elaborate theory suggests the presence of two factors driving interest rates. In this *two-factor model* the first factor is often referred to as a *level* factor that determines the level of all interest rates, while the second is interpreted as a *slope factor* determining the spreads between interest rates. A very simple example of this would correspond to the following scenario

$$\begin{pmatrix} R0_t \\ R1_t \\ R2_t \\ R5_t \\ R10_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1/4 \\ 1 & 2/4 \\ 1 & 3/4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^t u_{1i} \\ \sum_{i=1}^t u_{2i} \end{pmatrix} + \text{stationary process} + \text{initial value}, \tag{1.4}$$

where the two non-stationary factors are now $f_t^1 = \sum_{i=1}^t u_{1i}$ and $f_t^2 = \sum_{i=1}^t u_{2i}$.

[3] Explain that the scenario in (1.4) implies a set of cointegrating relations that are given by the cointegration matrix

$$\beta = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.5}$$

Are the interest rate spreads, e.g. $R0_t - R10_t$ or $R1_1 - R10_t$, stationary under this scenario?

The final theory considered here suggests the presence of three factors driving the interest rates. The new factor added in the *three-factor model* is often interpreted as a factor determining the *curvature* of the term structure, which, for a given level and slope, allows for a $\cup$-shape or a $\cap$-shape of the yield curve. A simple version of this idea would correspond to a scenario of the form

$$\begin{pmatrix} R0_t \\ R1_t \\ R2_t \\ R5_t \\ R10_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/4 & 1 \\ 1 & 2/4 & 2 \\ 1 & 3/4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^t u_{1i} \\ \sum_{i=1}^t u_{2i} \\ \sum_{i=1}^t u_{3i} \end{pmatrix} + \text{stationary process} + \text{initial value}, \tag{1.6}$$

with non-stationary factors $f_t^m = \sum_{i=1}^t u_{mi}$, $m = 1, 2, 3$. This final scenario implies a set of cointegrating relations given by

$$\beta = \begin{pmatrix} 1 & 0 \\ -2 & 0 \\ 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{pmatrix}. \tag{1.7}$$

[4] Assume that the *three-factor model* was the data generating process. What would happen, in theory, with the Granger representation and the cointegration structure if $R10_t$ was omitted from the analysis, i.e. if you looked at $z_t = (R0_t : R1_t : R2_t : R5_t)'$.

In the empirical analysis we want to confront the suggested theories with the data in order to understand the dynamics of the term structure of interest rates.

Regarding the *institutional setup* of your specific country, you are informed that the restrictions on the international flow of capital were permanently changed in November 2000. Commentators have argued that this liberalization could have changed the dynamics of the interest rates.

2 The Statistical Model

First we want to have a look at the time series for interest rates, and find a statistical model that represents the main features of the data.

[5] Perform a graphical analysis of the time series in the data vector in (1.1).
Comment on the time series behavior of the variables and look for potential indications of the validity of the one-, two-, and three-factor models presented above.

[6] Write a relevant vector autoregressive (VAR) model with k lags in level form and state the assumptions we maintain for the empirical model.
Argue, in particular, for your choice of deterministic terms in the VAR model.

[7] Based on the assumption of Gaussian error terms use sequential factorization to write the likelihood function of the VAR(k) model for the sample, $x_1, x_2, ..., x_T$, conditional on the initial values, $x_0, x_{-1}, ..., x_{-(k-1)}$.

[8] Estimate a VAR model for interest rates based on your preferred specification.
Test the maintained assumptions from question [6].
Modify and respecify the model until you have a satisfactory representation of the data. You may have to try different models to reach a satisfactory specification, but to save space you only need to outline the steps in your progress and present your final model. Note that it may not be possible to find a model that is acceptable in all directions, just do as well as you can.

3 The Cointegration Rank

Next step in the analysis is to determine the cointegration rank.

[9] Show how to derive the error correction form of the VAR model for your preferred model from Section 2. Write the characteristic polynomial for the error correction form and explain what the presence of unit roots implies for the parameters in the error correction form.

[10] Estimate the roots of the characteristic polynomial for your preferred empirical interest rate model.
What does that suggest in terms of the stationarity of interest rates?

[11] Determine the number of cointegrating relationships, r, taking all the available sources of information into account.
Explain, in particular, how you treat the deterministic terms in the likelihood ratio testing procedure, and make sure that you use the correct critical values.

[12] What does the cointegration rank imply for the relevance of the factor models in Section 1?

4 Testing Hypotheses

To learn about the dynamics of interest rates, we now want to test restrictions on the equilibrium structure and the error correction properties.

[13] Impose your preferred cointegration rank, r, on the error correction form of the VAR model and estimate the CVAR, $H(r)$ say, using maximum likelihood.
Comment, in detail, on the results. In particular, explain to what extend the parameters are identified.

[14] The likelihood function of the CVAR is maximized by solving a particular eigenvalue problem. Discuss the advantages of the eigenvalue approach compared to a more general approach, where the likelihood function of the model, $H(r)$, is maximized numerically with respect to the parameters, e.g. $\theta = \{\alpha, \beta, \Gamma_1, ..., \Gamma_{k-1}, \Omega\}$.

[15] Find the Granger representation corresponding to the CVAR for the interest rate data and comment on the results.

[16] Explain the test for *long-run exclusion* and perform the test for all variables.
If a variable is found to be long-run excludable, does that mean that the variable can be removed from the model altogether?

[17] Test for *weak exogeneity* of all variables with respect to the cointegrating parameters in β. Carefully explain what a weakly exogenous variable implies for the common trends of the model. Also explain the relationship between weak exogeneity and conditional models.

5 Identification

Now we want to impose restrictions on the cointegration space of the form

$$\beta = (\beta_1 : \beta_2 : ... : \beta_r) = (H_1\varphi_1 : H_2\varphi_2 : ... : H_r\varphi_r),$$

where H_j is a known design matrix and φ_j contains the free parameters to be estimated, $j = 1, 2, ..., r$.

[18] For the most relevant of the theoretical models in Section 1, state the design matrices, $H_1, H_2, ..., H_r$, and check that the structure is actually generically identifying.

[19] For the empirical model, impose just identifying restrictions on β. Your chosen just-identifying restrictions should be consistent with the relevant theoretical framework. Simplify the structure as much as possible inspired by the magnitude and significance of the estimated parameters and by the suggestions of the theoretical model.
For your preferred identified model, carefully explain the equilibrium structure and the error correction properties.

[20] Reconsider the Granger representation for the identified structure and interpret the results.
Explain which shocks have permanent effects and how they affect the variables.
How does this relate to the theoretical models in Section 1?

6 Final Questions: Extensions

[21] Explain the idea of the *Structural MA model* in CATS and how it can be used to estimate impulse-response functions.
Based on your preferred identified model for the interest rate data, suggest restrictions to identify the permanent structural shocks and perform an impulse response analysis.
Why is this approach *not* convenient for the identification of the level-, slope-, and curvature-factors in the theoretical framework?

[22] In question [7] you derived the likelihood function for the CVAR in the case of identical and independent Gaussian error terms, $\epsilon_t \mid x_{t-1}, ..., x_{t-k} \sim N(0, \Omega)$. Now assume that the error terms are characterized by autoregressive conditional heteroskedasticity (ARCH), such that $\epsilon_t \mid x_{t-1}, ..., x_{t-k} \sim N(0, \Omega_t)$, with conditional variance given by

$$\Omega_t = C + A\epsilon_{t-1}\epsilon_{t-1}'A',$$

where C is $(p \times p)$ and positive definite while A is $(p \times p)$ and unrestricted.
Modify the likelihood function in question [7] to allow for the conditional heteroskedasticity. This may not be an easy question, and any sensible progress is rewarded.